

# On computation of clustering coefficient in a class of random networks

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## Abstract

The random networks enriched with additional structures as metric and group-symmetry in background metric space are investigated. The important quantities like the clustering coefficient as well as the mean degree of separation in such networks are effectively computed with help of additional structures. Representative models are discussed in details.

**keywords:** random graph, clustering, degree of separation

**MCS:** 05Cxx

## 1 Introduction

It is well-known for about 40 years that the mean degree of separation of humans (the mean number of people needed to bind two randomly chosen persons via a chain of acquaintance) is surprisingly small, it takes the value about six. This was discovered for the first time by the American psychologists Stanley Milgram [1]. Since his days this very interesting phenomenon attracts the interest of scientists from various fields of research. As shown in [6], the small world phenomenon appears as a generic feature of many natural as well as artificial random graphs. To explore these properties of the random networks and to understand them one finds often helpful to use methods developed for statistical mechanics [5]. Generally, there is no hope to obtain exact results for main quantities describing random networks of real interest. However, under some additional condition one can build up simplified models obeying some kind of symmetries that allow for expressing crucial quantities (their approximations) in closed form. Methods of computation can be divided into two classes; one of them uses purely discrete point of view and the second one, which is much more effective if applicable, uses continuous limit of the network. We shall be oriented in the latest.

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The importance of the use of a continuum limit in various discrete random structures is surely undisputable because the exact discrete mathematics based methods have commonly bounded area of use. In current research the continuum limit methods are used widely, for the use in direct random graph theory see e.g. [2], an interesting result is achieved by these methods studying some hidden variables (variables that are not measured directly like degree of a node) in a kind of biological networks [3]. One of the most important areas of research where these continuum limits are used is the problem of phase transition in random structures (critical phenomena), for the review see e.g. [4] and further references therein. The main aim of the present paper is to show how to compute the relevant quantities like the clustering coefficient and the mean degree of separation of nodes in a given random network that is build up on a metric space or a space where we can define some oriented distance. The crucial assumption is that the network obeys some symmetry. The metric serves us to define the probability that two points (nodes, persons) are linked via a kind of acquaintance, and this probability is a given function of the distance between the nodes.

## 2 Definitions and basic properties of relevant quantities

Let us consider the number  $N$  of entities (people) represented as points lying in the metric space  $\Omega$  equipped with the distance  $d$ . The acquaintance (defined in a suitable way) between two people represented by the points  $A$  and  $B$  is recorded as a link between these points. With the help of the metric considered on  $\Omega$  we can construct a probability  $Q(A, B)$  that  $A$  knows  $B$  (and vice-versa, we consider always the reflexivity of knowing someone) as a function of the distance between  $A$  and  $B$ :

$$Q(A, B) = Q(d(A, B)). \quad (1)$$

The simplest quantity characterizing our network that is given by  $Q$  is the average number of acquaintance for any person  $A$ :

$$\mathcal{N} = \sum_B Q(A, B), \quad (2)$$

where the summation runs over all points of considered network. If one assumes that the function  $Q$  varies slowly on the scale of elementary distance in the network, then the above written sum can be approximated by the integral:

$$\mathcal{N} = \int_{\Omega} Q(x, y) dy \quad (3)$$

with proper measure  $dy$  on  $\Omega$ . (For example,  $\Omega$  can be considered to be an sub-manifold in some Euclidean space  $\mathbb{R}^n$  and then it is equipped naturally with both metric and volume form.)

Let  $b$  be the distance between the two considered points. Then the average degree of separation can be written as the mean value of the function (mean with respect to its argument  $S$ )  $P(S, b)$  that is the probability that the degree of separation of  $A$  and  $B$  at relative distance  $b$  equals  $S$ .

First of all,  $P(0, b)$  is given trivially by definition of  $Q$  itself:

$$P(0, b) = Q(b). \quad (4)$$

Other  $P(S, b)$ 's are to be computed nontrivially. And we are able to obtain analytical results only if some approximations in computing  $P(S, b)$  are applicable. Let us discuss  $P(1, b)$ . This means that there is just one person, say  $C$  separating the two chosen persons  $A$  and  $B$ . (Let us note that  $C$  is necessarily different from  $A$  as well as from  $B$ , otherwise  $A$  knows  $B$ .) With respect what has just been said we obtain the probability in question:

$$P(1, b) = \sum_C Q(A, C)Q(C, B)(1 - Q(A, B)). \quad (5)$$

Analogically we come at the expression for the probability that the separation of the two points at fixed distance is two:

$$P(2, b) = \sum_{C_1} \sum_{C_2} Q(A, C_1)Q(C_1, C_2)Q(C_2, B)(1 - Q(A, C_2)) \times \\ (1 - Q(B, C_1))(1 - Q(A, B)), \quad (6)$$

and we could continue to further separation indices in the same way. However, instead of doing this, we are going to describe an approximation we shall use in the future considerations. We assume that the probabilities  $Q(A, B)$  are small enough and therefore the product of two such probabilities (that is of the higher order of smallness) can be neglected in the expression (6) as well as in others  $P(S, b)$ 's. In this way one obtains instead of (6) its reduction:

$$P(2, b) \approx \sum_{C_1} \sum_{C_2} Q(A, C_1)Q(C_1, C_2)Q(C_2, B). \quad (7)$$

By means of derivation (3) from (2) we can write down for  $P(2, b)$  also the following integral formula

$$P(2, b) \approx \int_{\Omega \times \Omega} Q(A, x)Q(x, y)Q(y, B)(1 - Q(A, y))(1 - Q(x, B))(1 - Q(A, B))dxdy \\ \approx \int_{\Omega \times \Omega} Q(A, x)Q(x, y)Q(y, B)dxdy. \quad (8)$$

Obviously, the same can be done for all other  $P(S, b)$ 's with the similar result. The special structure of the integral formula (8) will allow for interesting expression for the essential quantity we shall turn now our attention to. The quantity

that is effectively and often used in analysis of random networks is the mean clustering coefficient  $\langle C \rangle$ , [7]. The quantity  $\langle C \rangle$  is the probability that (any) two acquaintances of given person  $A$  knows each other. By this definition,  $\langle C \rangle$  can be computed modifying equation (8) - namely we must identify  $A$  and  $B$  and avoid interchanging the positions of persons  $C_1$  and  $C_2$  that means to reduce the average number of triples by the factor  $1/2$ :

$$\frac{1}{2} \int_{\Omega \times \Omega} Q(A, x) Q(x, y) Q(y, A) dx dy.$$

To obtain  $\langle C \rangle$  we have to divide the above written average number of triples by the number of possible triples that is given by the fact that average number of acquaintances for any person is  $\mathcal{N}$ , so the normalizing constant we are looking for reads:  $\frac{\mathcal{N}(\mathcal{N}-1)}{2} \approx \frac{\mathcal{N}^2}{2}$ , where we have assumed  $\mathcal{N} \gg 1$ . Finally, the clustering coefficient is given by the following simple equation:

$$\langle C \rangle = \frac{1}{\mathcal{N}^2} \int_{\Omega \times \Omega} Q(A, x) Q(x, y) Q(y, A) dx dy. \quad (9)$$

The idea of the following is that the above written quantities can be expressed in approximate closed form using additional assumptions on symmetry of the metric space in use. We shall demonstrate this in details on a concrete example, however, the idea can be extended to wide range of symmetric spaces in the same way as generalized Fourier series are introduced.

### 3 One dimensional closed model

Let us consider the standard circle  $S^1$  (one dimensional sphere) as our metric space  $\Omega$ . This space can be considered as the orbit of a point under the action of the  $U(1)$  (or alternatively  $SO(2)$ ) group. The position of any person is then given by its polar coordinate  $\phi$  ( $\phi \in [-\pi, \pi]$ ) only. The probability  $Q$  is the function of one real variable by means of

$$Q(\phi_1, \phi_2) = Q(\phi_1 - \phi_2) = Q(|\phi_1 - \phi_2|). \quad (10)$$

The topology of the problem tells us that  $Q$  is a period function with period  $2\pi$ . This implies we can expand the function  $Q$  into its Fourier series in trigonometric functions in the mentioned interval  $[-\pi, \pi]$ :

$$Q(\phi) = \sum_{k=-\infty}^{\infty} a_k e^{ik\phi}.$$

Taking into account (10) (and that  $Q$  is real-valued) we have that the Fourier coefficients obey the equation:  $a_{-k} = a_k$ . As an example we shall compute the  $P(1, b)$  and  $P(2, b)$  functions in our case. With respect to eq. (8) and using orthogonality property of trigonometric basis we have after some algebra

$$P(1, b) = 2\pi R(1 - Q(b)) \sum_{m=-\infty}^{\infty} a_m^2 \cos(mb), \quad (11)$$

where the parameter  $R$  is introduced as the radius of the considered circle. This parameter will not enter the results because it serves only as a geometrical tool to keep the distance between any two neighbours equal to 1 if suitable. This simply means that the volume form is  $Rd\phi$ . Supposing the persons  $A$  and  $B$  are distanced enough ( $b \gg 1 \Rightarrow Q(b) \ll 1$ ) we can write instead of eq. (11) its simpler form:

$$P(1, b) \approx 2\pi R \sum_{m=-\infty}^{\infty} a_m^2 \cos(mb). \quad (12)$$

Using the same idea we can find also the probability  $P(2, b)$ :

$$\begin{aligned} P(2, b) = & (2\pi)^2 R^2 (1 - Q(b)) \left\{ \sum_{m=-\infty}^{\infty} a_m^3 e^{-imb} - 2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_m a_n a_{m+n}^2 e^{-imb} + \right. \\ & \left. \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} a_{m+n+p} a_{m+n} a_m a_n a_p e^{-i(m+p)b} \right\}. \end{aligned} \quad (13)$$

This expression is a little bit more complicated, assuming, in the same way as previously,  $b$  is large enough and neglecting the sums consisting of higher than third power of the Fourier coefficient we have the approximative expression for the probability  $P(2, b)$ :

$$P(2, b) \approx (2\pi)^2 R^2 \sum_{m=-\infty}^{\infty} a_m^3 e^{-imb} = (2\pi)^2 R^2 \sum_{m=-\infty}^{\infty} a_m^3 \cos(mb). \quad (14)$$

Thus, the important quantity as the clustering coefficient is expressed as follows immediately from eq. (13) putting  $b = 0$ :

$$\begin{aligned} \langle C \rangle = \frac{4\pi^2 R^2}{N^2} \left\{ \sum_{m=-\infty}^{\infty} a_m^3 - 2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_m a_n a_{m+n}^2 + \right. \\ \left. \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} a_{m+n+p} a_{m+n} a_m a_n a_p \right\} \approx \frac{4\pi^2 R^2}{N^2} \sum_{m=-\infty}^{\infty} a_m^3. \end{aligned} \quad (15)$$

Furthermore, one could easily generalize the results (12) and (14) to the probability  $P(k, b)$ , the result reads:

$$P(k, b) = (2\pi R)^k \sum_{n=-\infty}^{\infty} a_n^{k+1} \cos(nb). \quad (16)$$

Now we shall apply our theoretical results to a concrete example of primary probability function  $Q$  often discussed in literature in this context.

### Uniform distribution within a fixed radius

Now, the function  $Q$  will represent the uniform distribution of probability within a fixed radius (angle), namely:

$$Q(\phi) = \begin{cases} p & , \quad \phi \in [-\Phi, \Phi] \\ 0 & , \quad \phi \notin [-\Phi, \Phi] \end{cases} , \quad (17)$$

where  $\Phi$  is a parameter lying in the interval  $[0, \pi]$  and  $p$  is the probability of an acquaintance, therefore  $p \in [0, 1]$ . The mean number of acquaintances of any given person in this model is given by

$$\mathcal{N} = 2R \int_0^\Phi p d\phi = 2Rp\Phi.$$

In order to obtain the clustering coefficient we are to find the Fourier coefficients of the function (17) in the interval  $[-\pi, \pi]$ :  $a_n = \frac{1}{2\pi} \int_{-\pi}^\pi Q(\phi) e^{in\phi} d\phi = \frac{1}{\pi} \int_0^\pi Q(\phi) \cos(n\phi) d\phi$ ;

$$a_0 = \frac{1}{\pi} \int_0^\pi p d\phi = \frac{p\Phi}{\pi}, \quad a_n = \frac{1}{\pi} \int_0^\pi p \cos(n\phi) d\phi = \frac{p \sin(n\Phi)}{\pi n}.$$

With respect to (15) the clustering coefficient is given by:

$$\langle C \rangle = \frac{p}{\pi\Phi^2} \left[ \Phi^3 + 2 \sum_{n=1}^{\infty} \frac{\sin^3(n\Phi)}{n^3} \right]. \quad (18)$$

Special case arises when  $\Phi = \pi$ , i.e. when one considers uniform distribution of probability throughout whole circle. In such a situation we have trivial result:

$$\langle C \rangle(\Phi = \pi) = p. \quad (19)$$

The result (18) is plotted in the graph 1. The probabilities  $P(k, b)$  in this model are given by (16):

$$\begin{aligned} P(k, b) &= (2\pi R)^k \left[ \left( \frac{p\Phi}{\pi} \right)^{k+1} + 2 \sum_{n=1}^{\infty} \left( \frac{p \sin(n\Phi)}{\pi n} \right)^{k+1} \cos(nb) \right] = \\ &= \frac{p}{\pi} \left( \frac{\mathcal{N}}{\Phi} \right)^k \left[ \Phi^{k+1} + 2 \sum_{n=1}^{\infty} \frac{\sin^{k+1}(n\Phi)}{n^{k+1}} \cos(nb) \right], \end{aligned} \quad (20)$$

and for the maximally distanced persons:

$$P(k, \pi) = \frac{p}{\pi} \left( \frac{\mathcal{N}}{\Phi} \right)^k \left[ \Phi^{k+1} + 2 \sum_{n=1}^{\infty} \frac{\sin^{k+1}(n\Phi)}{n^{k+1}} (-1)^n \right]. \quad (21)$$

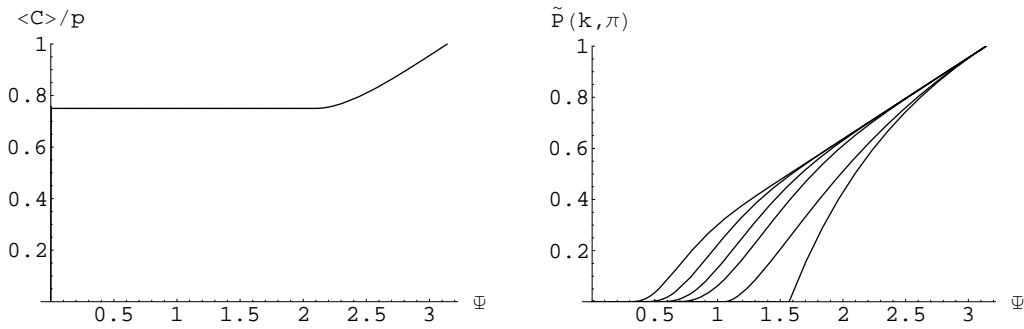


Figure 1: Left graph shows the plot of the fraction  $\langle C \rangle / p$  for the clustering coefficient given by (18). The fraction is nearly a constant (except the value at  $\Phi = 0$  that is, however, trivial) up to an angle a little bit greater than 2 and then starts to rise up to 1 as it should with respect to (19). Right one shows the plot of the functions:  $\tilde{P}(k, \pi) = P(k, \pi) / (\pi \mathcal{N}^k)$  in dependence on the angle  $\Phi$ . The curves are plotted (from the right to the left) for  $k = 1, 2, 4, 6, 10, 20$ . One can obviously identify critical values of the  $\Phi$ -parameter for  $\langle C \rangle / p$  as well as for  $\tilde{P}(k, \pi)$ .

## 4 Higher dimensional models

Now we shall briefly show how the previous consideration can be extended into higher dimensional case. For example, people living on Earth are to be modeled as, at least, two dimensional network (in fact one should incorporate also socio-logical dimension rather than geographical only).

There is an straightforward possibility how to generalize our one-dimensional example. Instead of  $S^1$  we can consider the  $K$ -dimensional tori  $T^K$ , i.e. the network with the topology

$$T^K := \underbrace{S^1 \times S^1 \times \dots \times S^1}_{K\text{-times}}.$$

The edges between any two neighbours have the unit length. Let  $(\phi_1, \phi_2, \dots, \phi_K)$  be the (periodic) coordinates. The probability that the persons  $A = (0, \dots, 0)$  and  $B = (\phi_1, \dots, \phi_K)$  knows each other is given by the functions of angular distances between the persons in each direction:

$$Q(A, B) = Q_1(\phi_1) \cdot Q_2(\phi_2) \cdots Q_K(\phi_K), \quad (22)$$

where any function  $Q_i$  plays the role of single function  $Q$  from the previous section. The prescription (22) for the probability  $Q$  allows for introducing an anisotropy in the model, if suitable. This situation in two dimension is shown in figure 2. Let us mention that this example shows we do not need to have metric space, the metric structure is replaced in this case by the capability to measure distances in separate directions.

The task to find the probabilities  $P(k, b_1, \dots, b_K)$  can be obviously solved in the same way as it was done in one dimensional case in previous section. The difference stands in only one detail that the single integrals (defining Fourier coefficients) are replaced by multiple ( $K$ -tuple) integrals defining the Fourier coefficients of functions defined on  $\underbrace{S^1 \times S^1 \times \dots \times S^1}_{K\text{-times}}$ . Of course, a practical difference

can appear, namely it is reasonable to expect the numerics in higher dimensional case will be more complicated. Moreover, one can use the same procedure for any homogenous space with help of generalized Fourier series. Especially, one can perform analogical computation also in the case when (at least in one dimension) the background space is unbounded - the only thing is to replace Fourier series by Fourier (generalized) transformation. Generalized orthogonality will allow for analogical expression of the clustering coefficient also in the case of non-compact symmetry group.

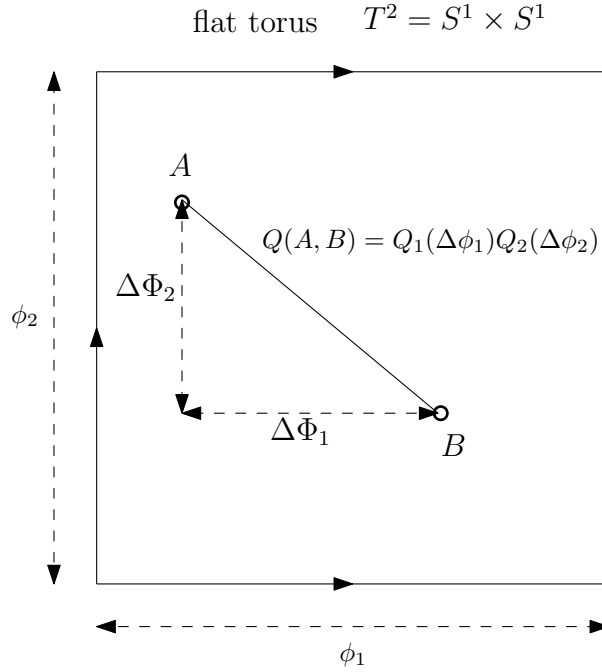


Figure 2: Two dimensional flat torus as two dimensional compact background space.

**Acknowledgement:** This work was supported partially by GRATEX Research Center and by the VEGA agency under project no. 1/3042/06 and by the UK grant 403/2007.

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